WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 5th Semester Examination, 2022-23

MTMADSE03T-MATHEMATICS (DSE1/2)

PROBABILITY AND STATISTICS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* questions from the rest

1. Answer any *five* questions from the following: $2\times5 = 10$

(a) Define a random experiment and event space.

(b) Consider events A and B such that $P(A)=\frac{1}{4}$, $P(B|A)=\frac{1}{2}$, $P(A|B)=\frac{1}{4}$. Find $P(\overline{A}|\overline{B})$

(c) Consider an experiment of rolling two dice. Let A be the event 'total is odd' and B be the event '6 on the first die'. Are A and B independent? Justify your answer.

(d) If $F(x)$ be the distribution function of a random variable X, then prove that $F(a)-\lim_{x\to a-0} F(x)=P(X=a)$

(e) The probability density function of a random variable X is $2x\cdot e^{-x^2}$ for $x>0$ and zero otherwise. Find the probability density of X^2.

(f) The joint probability density function of random variables X and Y is given by

$$f(x, y)=\begin{cases}2(x+y-3xy^2), & 0<x<1,\ 0<y<1\\ 0, & \text{elsewhere}\end{cases}$$

Find the marginal density functions of X and Y.

(g) Prove that $-1\leq\rho(X,Y)\leq 1$, the symbols having usual meaning.

(h) Find the characteristic function of a binomial (n, p) variate.

(i) Find the mean of a Poisson μ-variate.

2. (a) If A and B are two events such that $P(A)=P(B)=1$, then show that $P(A+B)=1$, $P(AB)=1$. 3

(b) A secretary writes four letters and the corresponding addresses on envelopes. If he inserts the letters in the envelopes at random irrespective of address, then calculate the probability that all the letters are wrongly placed. 5

3. Prove that the function $f(x)$ of a random variable X defined by $f(x)=\frac{1}{2}e^{-|x|}$, $-\infty<x<\infty$, is a possible probability density function and find the corresponding distribution function and the moment generating function and hence evaluate mean and variance. 8

4. (a) Define Poisson distribution. Prove that the sum of two independent Poisson variates having parameters μ_1 and μ_2 is a Poisson variates having parameter $\mu_1+\mu_2$. 4

(b) If θ be the acute angle between two regression lines, then prove that 4

$$\tan\theta=\frac{1-\rho^2}{\rho}\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

where σ_x and σ_y are standard deviations of the random variables X and Y respectively. What happens when $\rho=1$?

5. (a) For the binomial (n, p) distribution, prove that $\mu_{r+1}=p(1-p)\left[nr\mu_{r-1}+\frac{d\mu_r}{dp}\right]$ where μ_r is the rth central moment of the distribution. 5

(b) If $ax+by+c=0$ be the relation between x and y, find r_{xy}. 3

6. (a) The joint probability density function of two random variables X and Y is 5

$$f(x, y) = 8xy,\ 0\le x\le 1,\ 0\le y\le 1$$
$$= 0, \text{ elsewhere.}$$

Examine whether X and Y are independent. Also find the conditional probability density functions.

(b) Use Tchebycheff's inequality to show that for $n\ge 36$, the probability that in n throws of a fair die the number of sixes lies between $\frac{n}{6}-\sqrt{n}$ and $\frac{n}{6}+\sqrt{n}$ is at least $31/36$. 3

7. (a) Obtain the maximum likelihood estimate of θ on the basis of a random sample of size n drawn from a population whose probability density function is 5

$$f(x)=ce^{-x/\theta},\ 0\le x<\infty,$$

where c is constant and $\theta>0$.

(b) Two random variables X, Y have the least square regression lines with equations $3x+2y-26=0$ and $6x+y-31=0$. Find $E(X), E(Y)$ and $\rho(X, Y)$. 3

8. (a) A random variable X has probability density function $12x^2(1-x)$, $0<x<1$. Compute $P(|X-m|\ge 2\sigma)$ and compare it with the limit given by Tchebycheff's inequality, where m is the mean and σ is the standard deviation of the distribution. 5

(b) State and prove the law of large numbers. 3

9. (a) Find the sampling distribution of the statistic $Y=\frac{nS^2}{\sigma^2}$, where σ^2 is the population variance and $S^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$. 5

(b) Prove that the sample variance is a consistent estimate of the population variance but it is not an unbiased estimate of population variance. 3

10.(a) If $\{X_n\}_n$ is a sequence of independent variables such that each X_i has the same distribution with mean m and standard deviation σ, then show that $\frac{\bar{X}-m}{\sigma/\sqrt{n}}$ is asymptotically normal $(0, 1)$, where $\bar{X}=\frac{X_1+X_2+\cdots+X_n}{n}$. 4

(b) A point P is chosen at random on a circle of radius a and A be a fixed point on the circle. Find the expectation of the distance AP. 4

———×———